# Multirelations are predicate transformers

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In [2, 3], a *multirelation* from a set $X$ to a set $Y$ as defined as a binary relation between $X$ and the power set of $Y$. The power set of $Y$ can be identified with the set $\mathbb{P}(X)$ of predicates on $Y$. In other words, a multirelation is a predicate on the cartesian product $X \times \mathbb{P}(Y)$. Therefore, the domain of the multirelations from $X$ to $Y$ equals $\mathbb{P}(X \times \mathbb{P}(Y))$.

This little note is devoted to the observations, that the domain of the multirelations from $X$ to $Y$ is isomorphic to the domain of the predicate transformers from predicates on $Y$ to predicates on $X$, i.e. functions $\mathbb{P}(Y) \to \mathbb{P}(X)$, and that most of the operators that have been introduced in either domain also have been introduced in the other domain, often under the same names.

For readability and uniformity I treat both predicates and relations as subsets of the corresponding domains. A (monotonic) predicate transformer is then a (monotonic) subset transformer (monotonic with respect to set inclusion). I use the infix dot for function application. It associates to the left (as is usual in functional programming), to enable convenient currying.

If $M$ is a multirelation from $X$ to $Y$, let $wp.M$ be the predicate transformer from $Y$ to $X$ given by $wp.M.Q = \{x \in X \mid (x, Q) \in M\}$. Conversely, for $c : \mathbb{P}(Y) \to \mathbb{P}(X)$, we define the multirelation $|c|$ from $X$ to $Y$ to consist of the pairs $(x, Q) \in X \times \mathbb{P}(Y)$ with $x \in c.Q$. Then it holds that $|wp.M| = M$ since, for all $x$ and $Q$, we have

$$(x, Q) \in |wp.M| \quad \equiv \quad x \in wp.M.Q \quad \equiv \quad (x, Q) \in M \ ,$$

and, conversely, $wp.|c| = c$ since, for all $x$ and $Q$, we have

$$x \in wp.|c|.Q \quad \equiv \quad (x, Q) \in |c| \quad \equiv \quad x \in c.Q \ .$$

*Remark.* This looks suspiciously simple. So, it should be a special case of a more general triviality. Indeed, it is a special case of the well-known fact that, for arbitrary domains $U$, $X$, $W$, the domains $U \times X \to W$ and $U \to (X \to W)$ are in bijective correspondence (in functional programming, using this is known as "currying"). Using $\mathbb{P}(X) = (X \to W)$ when $W$ is the set of the booleans, we get a bijection between $\mathbb{P}(U \times X)$ and $U \to \mathbb{P}(X)$. Swapping $U$ and $X$ and taking $U = \mathbb{P}(Y)$, we obtain our bijection between $\mathbb{P}(X \times \mathbb{P}(Y))$ and $\mathbb{P}(Y) \to \mathbb{P}(X)$.

We turn to monotonicity. In [2, 3], a multirelation $M$ from $X$ to $Y$ is defined to be *up-closed* iff, for all $x \in X$ and all predicates $Q \subseteq Q'$ on $Y$, we have that $(x, Q) \in M$ implies $(x, Q') \in M$. In terms of $wp$, this says that $x \in wp.M.Q$ implies $x \in wp.M.Q'$, i.e., that $wp.M.Q \subseteq wp.M.Q'$. This shows that multirelation $M$ is up-closed if and only if $wp$.M is monotonic.

Angelic and demonic choice of multirelations are union and intersection. It is straightforward to verify that they correspond directly to the disjunction and conjunction of the predicate transformers, as considered e.g. in [1]. Angelic and demonic refinement can be defined in the usual way by means of the corresponding choice operators, and therefore also directly correspond to the natural notions. Duality of multirelations corresponds to conjugacy of predicate transformers [1].

The composition of up-closed multirelations $(M;N)$ satisfies

$$
\begin{array}{ll}
 & (x,Q) \in (M;N) \\
\equiv & \{ \text{ definition in } [2,3] \ \} \\
 & (\exists\, P : (x,P) \in M \ \wedge \ (\forall\, y \in P : (y,Q) \in N)) \\
\equiv & \{ \text{ definition of } \mathit{wp}.N \ \} \\
 & (\exists\, P : (x,P) \in M \ \wedge \ P \subseteq \mathit{wp}.N.Q) \\
\equiv & \{ \text{ use that } M \text{ is up-closed } \} \\
 & (x, \mathit{wp}.N.Q) \in M \\
\equiv & \{ \text{ definition of } \mathit{wp}.M \ \} \\
 & x \in \mathit{wp}.M.(\mathit{wp}.N.Q) \\
\equiv & \{ \text{ definition of composition } \} \\
 & x \in (\mathit{wp}.M \circ \mathit{wp}.N).Q \ .
\end{array}
$$

Since the first line is equivalent to $x \in \mathit{wp}.(M;N).Q$, this shows that our function *wp* transforms sequential composition of multirelations to functional composition of predicate transformers.

We thus see that the up-closed multirelations form a very adequate model for the monotonic predicate transformers, and vice versa. The paper [2] does not mention this. The paper [3] gives our function *wp* from multirelations to predicate transformers, but not its inverse. Strictly speaking, it only introduces predicate transformers $g_M$ with $g_M(Q) = \mathit{wp}.M.Q$, but it seems not to treat $g_M$ as a function of $M$ and therefore misses the observation that function *wp* (or $g$) is a bijection with an easily expressable inverse.

The methodological question remains: shall we prefer predicate transformers or multirelations, or perhaps use both models? Predicate transformers have a more natural composition. The lattice operators (union, intersection, order, top, and bottom) are equally simple in both models. One can argue that the multirelation model is more intuitive or more natural. The latter model may also suggest some useful specification mechanisms. In any case, the correspondence is illuminating.

## References


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