On quadratic pruning of IMA

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Introduction

Let us first recall the setting and the main definitions. We consider d-dimensional binary images, where $d \geq 2$. A d-dimensional binary image is a pair (A, X), where A is a rectangular box in the grid $\mathbb{Z}^d$ and X is a subset of A, which is called the foreground. For practical purposes, we are more interested in the background B, which is the complement $A \setminus X$, possibly augmented with some points outside A.

We regard $\mathbb{Z}^d$ as embedded in $\mathbb{R}^d$ and use the Euclidean norm on $\mathbb{R}^d$, which is the nonnegative square root of $||x||^2 = \sum_{i=1}^{d} x_i^2$. Let $E = \{e \in \mathbb{Z}^d \mid ||e|| = 1\}$, the set of integer unit vectors.

For every point $p \in \mathbb{R}^d$, the *distance transform* $dt(p)$ is defined as the minimal squared distance $||p - x||^2$ where x ranges over B. The *feature transform sets* are defined by $FT(p) = \{x \in B \mid ||p - x||^2 = dt(p)\}$. A *feature transform function* is a function $A \to B$ that, for each foreground point p, chooses an element of $FT(p)$, i.e. a background point where the minimum is reached. So, we have $||p - ft(p)||^2 = dt(p)$.

Once a feature transform function *ft* has been chosen, the integer medial axis *IMA* is defined (cf. [2]) to consist of the points $p \in A$ such that for some $e \in E$ we have $||ft(p) - ft(p+e)|| > 1$ and $||m - ft(p)|| \geq ||m - ft(p+e)||$ where $m = p + \frac{1}{2}e$.

If the background consists of isolated dots, i.e. $p + e \notin B$ whenever $p \in B$ and $e \in E$, then *IMA* gives the boundaries of the Voronoi cells. More generally, for 2-dimensional images, we conjecture that, if the foreground is bounded and 8-connected, *IMA* is also 8-connected.

Constant pruning and linear pruning

For many images, both natural and artificial, *IMA* contains many "unwanted" points: points that a human observer would not regard as belonging to a natural skeleton.

In order to get a "better" skeleton, we proposed in [2] to modify the definition of *IMA* as follows. For any $\gamma \in \mathbb{N}$, let $IMA(\gamma)$ consist of the points $p \in A$ such that for some $e \in E$ we have $||ft(p) - ft(p+e)|| > \gamma$ and $||m - ft(p)|| \geq ||m - ft(p+e)||$ where $m = p + \frac{1}{2}e$.

In this way, *IMA* points are pruned for which the closest background points are too close together. Let us call this method *constant pruning*. For many practical images, one can find a value for γ such that $IMA(\gamma)$ is an acceptable skeleton. The best value for γ depends on the coarseness of the image. On the one hand, a small value of γ leads to "unwanted" skeleton points due to discretization of the background B. On the other hand, if one wants a skeleton that traverses a channel with a width of k pixels, one needs to take $\gamma < k^2$.

For many images, one would want to apply a pruning constant γ that varies over the different regions of the image, namely small where dt is small and large where dt is large.

This suggests to take γ proportional to dt. More precisely, for any $\gamma \in \mathbb{N}$, let $IMA[\gamma]$ consist of the points $p \in A$ such that for some $e \in E$ we have $||ft(p) - ft(p+e)||^2 > \gamma \cdot dt(p)$ and $||m - ft(p)|| \geq ||m - ft(p+e)||$ where $m = p + \frac{1}{2}e$. Roughly speaking, $ft(p)$ and $ft(p+e)$ form the equal legs of an isosceles triangle with top angle φ where $\sin(\frac{1}{2}\varphi) < \frac{1}{2}\sqrt{\gamma}$. In other words, the parameter γ here plays the same role as the bisector angle considered in [1]. Let us call this method *linear pruning.*

Towards quadratic pruning

Pruning the skeleton of ponts with a bisector angle less the a fixed angle φ_0 (or equivalently, taking γ proportional to dt) has the following disadvantage. Assume (in 2D) that the background is the union of two halfplanes with an angle $\pi-\varphi$ where $\varphi < \varphi_0$. Then every point of the bisectrix of the angle has two feature transform points on the boundary lines of the halfplanes with a bisector angle φ. Since $\varphi < \varphi_0$, the whole bisectrix is eliminated from the skeleton. This is unsatisfactory, since this bisectrix is the real medial axis *RMA* as defined in [2].

The remedy we propose is to let γ grow slower than dt, roughly speaking proportional to the square root of dt. In the case of an obtuse angle between two halfplanes, this would have the effect that only the *IMA* points close to the top are pruned from the bisectrix. The choice to take a factor like the square root of dt is justified as follows.

A *halfspace* of $\mathbb{R}^d$ is a subset of the form $H = \{x \in \mathbb{R}^d \mid (u,x) \geq c\}$ for some $u \in \mathbb{R}^d$ and some $c \in \mathbb{R}$. Here $(\_,\_)$ is used to denote the standard inner product of $\mathbb{R}^d$. An integral halfspace is an intersection $H \cap \mathbb{Z}^d$ where H is a halfspace. We now want to prune our definition of *IMA* in such a way that the *IMA* is empty whenever the background is an integral halfspace, and not more than necessary for this purpose.

In order to get a good pruning value, we need to know, for any pair of neighbouring grid points, the maximal distance between feature transform points in any halfspace, expressed in something like their distance transforms. We do not completely know the answer, but we have a reasonably supported guess and a fully supported partial answer. We first present the guess.

Conjecture 1 *Assume that B is an integral halfspace. Let p, $q \in \mathbb{Z}^d$ and $x \in FT(p)$ and $y \in FT(q)$. Then $||x-y||^2 < 2\cdot(p-q, x-y) + ||x-p+y-q|| + 1$.*

The critical point is the occurence of a square to the left of the inequality sign and the absence of squares to the right. We have verified the conjecture for $||p-q|| = 1$ in 2D for images of 1000 by 1000, and in 3D on images of 100 by 100 by 200, for several settings of B. The critical value to compute is

$$crit = ||x-y||^2 - 2\cdot(p-q, x-y) - ||x-p+y-q|| \ .$$

With $||p-q|| = 1$, the maximal value for *crit* that we found in 2D-images of 1000 by 1000 with 99 different slopes of the halfplanes was $9-\sqrt{74} \approx 0.4$. This value was obtained with $p = (5,4)$, $q = (4,4)$, $x = (2,0)$, $y = (0,3)$. In 3D, the largest *crit* value found was 0.497, obtained with $p = (0,0,145)$, $q = (0,0,144)$, $x = (0,17,1)$, and $y = (1,0,0)$.

The conjecture was inspired by the following example with $p = q$.

Example. Consider the two-dimensional grid with $p = q = (0,0)$ and $x = (0, n+1)$ and $y = (m,n)$ with m, $n \in \mathbb{N}$. Assume that B consists of the grid points north of the line through x and y. Then $x \in FT(p)$ and $y \in FT(p)$ means that $||x|| = ||y||$, that is $(n+1)^2 = m^2 + n^2$, that is $2n+1 = m^2$.

In this situation we have $||x-y||^2 = m^2+1$ and $(p-q, x-y) = 0$ and $||x-p+y-q|| = ||(m, 2n+1)|| = m^2\cdot\sqrt{1+m^{-2}} > m^2$. This implies that $crit < 1$. The maximal value of *crit* we obtain in this way is $2-\sqrt{2}$. For large m, *crit* approximates $\frac{1}{2}$. □

In view of Conjecture 1, we define *IMAQ* to consist of the points $p \in A$ such that for some $q \in A$ we have $||p-q|| = 1$ and $||ft(p) - ft(q)|| > 1$ and

$$(0) \qquad ||ft(p) - ft(q)||^2 \geq 2 \cdot (p-q, ft(p) - ft(q)) + ||ft(p) + ft(q) - p - q|| + 1$$

and $||m - ft(p)|| \geq ||m - ft(q)||$ where $m = \frac{1}{2}(p+q)$. The definition immediately implies that $IMAQ \subseteq IMA$.

The application of formula (0) in the definition of *IMAQ* is called *quadratic pruning*. The pruning precludes *IMAQ* to traverse a very narrow channel, like a single Manhattan path. *IMAQ* does traverse a channel, however, if it is wide enough for two pawns to walk abreast. *IMAQ* is not necessarily connected. For example, in 2D, if the image consists of the points $(0,0)$, $(4,0)$, $(0,20)$, and $(4,20)$, *IMAQ* consists of three segments: the short segments of the points $(2,t)$ with $|t| \leq 4$ and $|t-20| \leq 4$, and the long segment of the points $(t,10)$ with $|t-2| \leq 181$.

In an attempt to prove Conjecture 1, we obtained the following partial result. Recall that the infinity norm on $\mathbb{R}^d$ is given by $||x||_\infty = \max_i |x_i|$. This norm is topologically equivalent to the Euclidean norm since we have $||x||_\infty \leq ||x|| \leq \sqrt{d} \cdot ||x||_\infty$.

Theorem 2 *Assume that B is an integral halfspace in $\mathbb{Z}^d$. Let p, q, x, $y \in \mathbb{Z}^d$ with $x \in FT(p)$ and $y \in FT(q)$. Then $||x-y||^2 \leq 2 \cdot (p-q, x-y) + 2 \cdot ||x-p+y-q||_\infty + d$.*

Proof. Since the problem is translation invariant, we may assume that $p = 0$. Then we have $x \in FT(0)$, $y \in FT(q)$.

Let B be given by $B = \{z \in \mathbb{Z}^d \mid (u,z) \geq c\}$ for some nonzero vector $u \in \mathbb{R}^d$ and some $c \in \mathbb{R}$. The midpoint $w = \frac{1}{2}(x+y)$ satisfies $(u,w) \geq c$. It is not necessarily in B since its coordinates belong to $\mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$. We now consider vectors f with $w+f \in B$. Since $w+f \in B$ and $x \in FT(0,B)$ and $y \in FT(q,B)$, we have $||x|| \leq ||w+f||$ and $||y-q|| \leq ||w+f-q||$. This is equivalent to

$$4(x,x) \leq (x+y+2f, x+y+2f) \ ,$$
$$4(y-q, y-q) \leq (x+y+2f-2q, x+y+2f-2q) \ .$$

Addition of these two inequalities yields

$$2(x,x) + 2(y,y) \leq 4(x,y) + 4(q, y-x) + 8(f, x+y+f-q) \ .$$

Division by 2 gives

$$||x-y||^2 \leq 2(q, y-x) + 4(f, x+y+f-q) \ .$$

It remains to prove that there is a vector $f \in \mathbb{R}^d$ with $w+f \in B$ and $4(f, x+y-q+f) \leq 2 \cdot ||x+y-q||_\infty + d$. Here $w = \frac{1}{2}(x+y)$. We write $v = x+y-q$. Then we have $2w = x+y = v+q$.

Since $B = \{z \in \mathbb{Z}^d \mid (u,z) \geq c\}$ and x, $y \in B$, it suffices to find f with $w+f \in \mathbb{Z}^d$ and $(u,f) \geq 0$ and $4(f, v+f) \leq 2 \cdot ||v||_\infty + d$. We now put $g = 2f$. It then suffices to find a vector g with $v+q+g \in 2\mathbb{Z}^d$ and $(u,g) \geq 0$ and $(g, 2v+g) \leq 2 \cdot ||v||_\infty + d$. This conjunction is equivalent to $v+g \in q+2\mathbb{Z}^d$ and $(u, v+g) \geq (u,v)$ and $||v+g||^2 - ||v||^2 \leq 2 \cdot ||v||_\infty + d$. Putting $z = v+g$, it remains to find a vector $z \in q+2\mathbb{Z}^d$ with $||z||^2 \leq ||v||^2 + 2 \cdot ||v||_\infty + d$ and $(u,z) \geq (u,v)$.

Let $C(v) = \{z \in \mathbb{R}^d \mid ||z||^2 \leq ||v||^2 + 2 \cdot ||v||_\infty + d\}$. It suffices to prove that $(u,z) \geq (u,v)$ for some $z \in (q+2\mathbb{Z}^d) \cap C(v)$. If $(u,z) < (u,v)$ for all $z \in (q+2\mathbb{Z}^d) \cap C(v)$, then v is not in the convex hull of $(q+2\mathbb{Z}^d) \cap C(v)$. It therefore suffices to prove the next lemma. □

Lemma 3 *Let v, $q \in \mathbb{Z}^d$. Then v is in the convex hull of $(q+2\mathbb{Z}^d) \cap C(v)$.*

Proof. By symmetry, we may assume that the coordinates of v are all positive and that they are descending. So, we have $v = (v_1, \ldots, v_d)$ with $v_1 \geq \ldots \geq v_d \geq 0$. In order to determine points in $(q + 2\mathbb{Z}^d) \cap C(v)$ that have v in their convex hull, we define J to be the set of indices j with $v_j \not\equiv q_j (\mathrm{mod}\ 2)$, that is, $J = \{j \mid v_j \notin q_j + 2\mathbb{Z}\}$. Assume that J has k elements. Note that $k \leq d$.

Let sequence t be the increasing enumeration of the set J that starts at 0, i.e. $J = \{t(i) \mid 0 \leq i < k\}$ and $t(0) < \ldots < t(k-1)$. Let $e_1, \ldots, e_d$ in $\mathbb{Z}^d$ be the standard basis of $\mathbb{R}^d$. Now consider the alternating sum $g = \sum_i (-1)^i e_{t(i)}$. Then we have $v \pm g \in q + 2\mathbb{Z}^d$. The vector v is obviously in the convex hull of the two vectors $v \pm g$. In order to prove that the vectors $v \pm g$ are in $C(v)$, we compute

$$||v \pm g||^2 - ||v||^2 = ||g||^2 \pm 2(g, v) = k \pm 2 \cdot \textstyle\sum_i (-1)^i v_{t(i)} \ .$$

The coordinates $v_{t(i)}$ are positive and descending. This implies that

$$||v \pm g||^2 - ||v||^2 = k \pm 2 \cdot \textstyle\sum_i (-1)^i v_{t(i)} \leq k + 2 \cdot v_{t(0)} \leq d + 2 \cdot ||v||_\infty \ .$$

This implies that both vectors $v \pm g$ are in $C(v)$, thus proving that v is in the convex hull of $(q + 2\mathbb{Z}^d) \cap C(v)$. □

Theorem 2 indeed gives a linear bound for the square $||x - y||^2$. The theorem might suggest to replace the righthand side of equality (0) by

$$2 \cdot (p - q, ft(p) - ft(q)) + 2 \cdot ||ft(p) + ft(q) - p - q||_\infty + d \ .$$

Such a definition is unsatisfactory, however, since it deletes more points of *IMA* than necessary.

It may be useful for the understanding of the conjecture and the theorem to record the following additional result.

Lemma 4 *Assume that B is an integral halfspace in $\mathbb{Z}^d$. Let p, q, x, $y \in \mathbb{Z}^d$ with $x \in FT(p)$ and $y \in FT(q)$. Then $(p - q, x - y) \geq 0$.*

Proof. Again we may assume that $p = 0$. Then we have $||x|| \leq ||y||$ and $||y - q|| \leq ||x - q||$. This implies $(x, x) \leq (y, y)$ and $(y - q, y - q) \leq (x - q, x - q)$. Adding these inequalities gives $2(q, x - y) \leq 0$ and hence $(p - q, x - y) \geq 0$. □

Remarks. In the experiments and in the applications we are not using arbitrary points $x \in FT(p)$ and $y \in FT(q)$, but $ft(p)$ and $ft(q)$ which are chosen in $FT(p)$ and $FT(q)$ in a systematic way. This may have influenced the experiments.

In the setting of the theorem, the value of $||x - y||^2 - 2 \cdot (p - q, x - y) - ||x - p + y - q||_\infty$ seems to be unbounded. So the factor 2 before the infinitity norm cannot be eliminated.

References

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