Universally Distributive Ordered Sets
— several known results —

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Abstract. Complete distributivity of ordered sets is expressed with choice functions and therefore often requires the Axiom of Choice (AC). It is known that AC can be avoided. We use the term universal distributivity for the version of complete distributivity that avoids AC. It is proved here that universal distributivity is self-dual: the dual of a universally distributive ordered set is itself universally distributive. It is also proved that every complete chain is universally distributive, and that every product of universally distributive ordered sets is universally distributive. Finally, two apparently simpler characterizations of universal distributivity are provided. All these results are known, but not so easy to find. It should also be known that the power set of an arbitrary set is universally distributive.

1 Introduction

A personal note of mine, dated January 1987, refers to [BD74](p. 232) for the theorem that complete distributivity of complete lattices is self-dual. The book was in the mathematics library in Austin, Texas, so I cannot consult it conveniently. The Wikipedia item for complete distributivity of 14 November 2006 gives a formulation of complete distributivity that is independent of the Axiom of Choice. This formulation is equivalent to universal distributivity as defined below. Since the item lacks proofs, or references that are easily consulted, I treat the matter here from scratch.

The present text is a minor extension of a text written January 2007 and extended in January 2009.

2 Universal Distributivity

Following [BW90], we use the term order and ordered set where other authors use partial order and partially ordered set (or poset). Recall that an ordered set X is called *complete* if every subset K of X has a least upper bound (denoted $\bigvee K$) and a greatest lower bound (denoted $\bigwedge K$) in X.

For any set X, we write $\mathbb{P}.X$ to denote its power set, i.e., the set of its subsets. Let X be a complete ordered set. For sets of subsets S, $T \in \mathbb{P}.(\mathbb{P}.X)$, we are interested in relating $\bigvee\{\bigwedge A \mid A \in S\}$ and $\bigwedge\{\bigvee B \mid B \in T\}$. We first note that

$$
\begin{array}{ll}
 & \bigvee\{\bigwedge A \mid A \in S\} \le \bigwedge\{\bigvee B \mid B \in T\} \\
\equiv & \{ \text{ definitions of } \bigvee \text{ and } \bigwedge \ \} \\
 & \forall\, A \in S, B \in T :\ \bigwedge A \le \bigvee B \\
\Leftarrow & \{\ x \in A \cap B \text{ implies } \bigwedge A \le x \le \bigvee B\ \} \\
 & \forall\, A \in S, B \in T :\ A \cap B \neq \emptyset\ .
\end{array}
$$

This proves that

$$
\begin{array}{lll}
(0) & & (\forall\, A \in S, B \in T : A \cap B \neq \emptyset) \\
 & & \Rightarrow \quad \bigvee\{\bigwedge A \mid A \in S\} \le \bigwedge\{\bigvee B \mid B \in T\}\ .
\end{array}
$$

We are now interested in a symmetric condition on S and T that implies equality in the consequent of (0).

For any $S \in \mathbb{P}.(\mathbb{P}.X)$, we define the *up-closure* $up.S$ and the opponent $S^{\#}$ by

$$
\begin{aligned}
&up.S = \{K \in \mathbb{P}.X \mid \exists\, A \in S : A \subseteq K\}\ ,\\
&S^{\#} = \{B \in \mathbb{P}.X \mid \forall\, A \in S : A \cap B \neq \emptyset\}\ .
\end{aligned}
$$

The antecedent of (0) is equivalent to $T \subseteq S^{\#}$, and also to $S \subseteq T^{\#}$. The relevance of the up-closure is that $\{\bigwedge A \mid A \in up.S\} = \{\bigwedge A \mid A \in S\}$ so that

(1) $$\bigvee\{\bigwedge A \mid A \in up.S\} = \bigvee\{\bigwedge A \mid A \in S\}\ .$$

If $S \subseteq T^{\#}$ then $S \subseteq up.S \subseteq T^{\#}$. This may suggest $up.S = T^{\#}$ as a condition to imply equality in the consequent of (0). Indeed, this is a symmetric condition, since we have, with K ranging over $\mathbb{P}.X$,

$$
\begin{aligned}
& up.S = T^{\#}\\
\equiv\ & \{\ \text{definitions}\ \}\\
& \forall\, K : (\exists\, A \in S : A \subseteq K) \equiv (\forall\, B \in T : B \cap K \neq \emptyset)\\
\equiv\ & \{\ \text{negate both operands of} \equiv\ \}\\
& \forall\, K : (\forall\, A \in S : A \cap (X \setminus K) \neq \emptyset) \equiv (\exists\, B \in T : B \subseteq (X \setminus K))\\
\equiv\ & \{\ \text{replace } X \setminus K \text{ by } K\ \}\\
& \forall\, K : (\forall\, A \in S : A \cap K \neq \emptyset) \equiv (\exists\, B \in T : B \subseteq K)\\
\equiv\ & \{\ \text{definitions}\ \}\\
& S^{\#} = up.T\ .
\end{aligned}
$$

Let us define a pair of sets S and T to be *entwined* if $up.S = T^{\#}$, or equivalently if $S^{\#} = up.T$. We define ordered set X to be *universally distributive* if it is complete and, for every entwined pair S and T, we have

(2) $$\bigvee\{\bigwedge A \mid A \in S\} = \bigwedge\{\bigvee B \mid B \in T\}\ .$$

It is easy to verify that $up.(S^{\#}) = S^{\#}$. Therefore S and $S^{\#}$ always are entwined. If X is universally distributive, we therefore always have

$$
\begin{aligned}
&\bigvee\{\bigwedge A \mid A \in S\} = \bigwedge\{\bigvee B \mid B \in S^{\#}\}\ ,\ \text{ and}\\
&\bigwedge\{\bigvee A \mid A \in S\} = \bigvee\{\bigwedge B \mid B \in S^{\#}\}\ .
\end{aligned}
$$

Since entwinedness is symmetric, universal distributivity is self-dual:

Theorem 1. *Let X be a universally distributive ordered set. Then the dual ordered set $(X, \leq^{\circ})$ given by $x \leq^{\circ} x' \equiv x' \leq x$ is also universally distributive.*

Classically, a complete ordered set X is called *completely distributive*, if for every $T \in \mathbb{P}.(\mathbb{P}.X)$, we have

(3) $$\bigwedge\{\bigvee B \mid B \in T\} = \bigvee_{f \in \prod T} \bigwedge_{B \in T} f.B\ .$$

Here $\prod T = \prod_{B \in T} B$ is the set of choice functions of T. Using $f^{\dagger}$ for the direct image function, the righthand side of (3) can be reformulated as

$$
\begin{aligned}
& \bigvee_{f \in \prod T} \bigwedge_{B \in T} f.B\\
=\ & \bigvee_{f \in \prod T} \bigwedge f^{\dagger}.T\\
=\ & \bigvee\{\bigwedge A \mid A \in T'\}\ \text{ where }\ T' = \{f^{\dagger}.T \mid f \in \prod T\}.
\end{aligned}
$$

Note that $T' \subseteq T^{\#}$ since $f.B \in f^{\dagger}.T \cap B$ for any $f \in \prod T$ and $B \in T$. Using (0), we get

$$\bigvee\{\bigwedge A \mid A \in T'\} \leq \bigvee\{\bigwedge A \mid A \in T^{\#}\} \leq \bigwedge\{\bigvee B \mid B \in T\}\ .$$

This shows that complete distributivity of X implies

$$\bigwedge\{\bigvee B \mid B \in T\} = \bigvee\{\bigwedge A \mid A \in T^{\#}\} \ .$$

If S and T are entwined, this implies formula (2) because of (1). This proves that complete distributivity implies universal distributivity.

We assume validity of the Axiom of Choice to prove the converse implication. Let T be given. For every $A \in T^{\#}$, the Axiom of Choice implies that there is some $f \in \prod_{B\in T}(A \cap B)$; this f satisfies $f \in \prod T$ and $f^{\dagger}.T \subseteq A$. Therefore, $T^{\#}$ is contained in the up-closure of T'. Since also $T' \subseteq T^{\#}$, it follows that $\{\bigwedge A \mid A \in T^{\#}\} = \{\bigwedge A \mid A \in T'\}$. Therefore, formula (2) with $S = T^{\#}$ implies $\bigwedge\{\bigvee B \mid B \in T\} = \bigvee\{\bigwedge A \mid A \in T'\}$, i.e., complete distributivity of X. To summarize, we have

Theorem 2. *(a) Any completely distributive complete ordered set is universally distributive.*
(b) The Axiom of Choice implies that any universally distributive ordered set is completely distributive.

3 Universal Distributivity of Chains

Recall that an ordered set X is called a *chain* if $x \leq x'$ or $x' \leq x$ for all x, $x' \in X$.

Theorem 3. *Let X be a complete chain. Then X is universally distributive.*

Proof. Let $T \in \mathbb{P}.(\mathbb{P}.X)$. Put $a = \bigvee\{\bigwedge A \mid A \in T^{\#}\}$ and $b = \bigwedge\{\bigvee B \mid B \in T\}$. It suffices to prove that $a = b$. Formula (0) implies $a \leq b$.

We therefore assume $a < b$ and aim at a contradiction. We first claim that there are a' and $b' \in X$ with the properties $a' < b$ and $a < b'$ and

$$(4) \qquad \forall\, z \in X : z < b' \ \Rightarrow\ z \leq a' \ .$$

There are two cases. If there is an element w with $a < w < b$, we can choose $a' = b' = w$. Otherwise, we can choose $a' = a$ and $b' = b$ since X is a chain.

We now observe

$$\begin{array}{ll}
 & \bigvee\{\bigwedge A \mid A \in T^{\#}\} = a \\
\Rightarrow & \{ \text{ we have } \bigwedge\{u \mid b' \leq u\} = b' > a \ \} \\
 & \{u \mid b' \leq u\} \notin T^{\#} \\
\equiv & \{ \text{ definition } T^{\#} \ \} \\
 & \exists\, B \in T : B \cap \{u \mid b' \leq u\} = \emptyset \\
\equiv & \{\ X \text{ is a chain } \} \\
 & \exists\, B \in T : \forall\, u \in B : u < b' \\
\Rightarrow & \{ \ (4) \ \} \\
 & \exists\, B \in T : \forall\, u \in B : u \leq a' \\
\Rightarrow & \{ \text{ some } B \text{ has } \bigvee B \leq a' \ \} \\
 & b = \bigwedge\{\bigvee B \mid B \in T\} \leq a' < b \ ,
\end{array}$$

which is a contradiction. □

Corollary 1. *The ordered set $\mathbb{B}$ of the Booleans with* false $<$ true *is universally distributive.*

4 Products of Universally Distributive Posets

Let $X = \prod_{i \in I} X.i$ be a product of a family $(i \in I : X.i)$ of ordered sets $X.i$, equipped with the argumentwise order: $x \leq y \equiv (\forall\, i : x.i \leq y.i)$. Recall that least upper bounds ($\bigvee$) and greatest lower bounds ($\bigwedge$) can be taken argumentwise. So, if all ordered sets $X.i$ are complete, the product X is complete.

Theorem 4. *Let $(i \in I : X.i)$ be a family of universally distributive ordered sets. Then the product $X = \prod_{i \in I} X.i$ is universally distributive.*

Proof. It suffices to observe that, for any entwined pair S and $T \in \mathbb{P}.(\mathbb{P}.(X))$, we have

$$
\begin{array}{ll}
 & \bigvee\{\bigwedge A \mid A \in S\} = \bigwedge\{\bigvee B \mid B \in T\} \\
\Leftarrow & \{ \text{ extensionality, take arbitrary } i \in I \ \} \\
 & \bigvee\{\bigwedge A \mid A \in S\}.i = \bigwedge\{\bigvee B \mid B \in T\}.i \\
\equiv & \{ \ \bigvee \text{ and } \bigwedge \text{ are taken argumentwise } \} \\
 & \bigvee\{\bigwedge\{f.i \mid f \in A\} \mid A \in S\} = \bigwedge\{\bigvee\{g.i \mid g \in B\} \mid B \in T\} \\
\Leftarrow & \{ \ X.i \text{ is universally distributive } \} \\
 & \{\{f.i \mid f \in A\} \mid A \in S\} \text{ and } \{\{g.i \mid g \in B\} \mid B \in T\} \text{ are entwined.}
\end{array}
$$

This entwinedness in $X.i$ is proved by observing that, for any subset K of $X.i$,

$$
\begin{array}{ll}
 & K \in up.\{\{f.i \mid f \in A\} \mid A \in S\} \\
\equiv & \{ \text{ definition } up \ \} \\
 & \exists\, A \in S : \{f.i \mid f \in A\} \subseteq K \\
\equiv & \{ \text{ calculus } \} \\
 & \exists\, A \in S : A \subseteq \{f \in X \mid f.i \in K\} \\
\equiv & \{ \text{ definition } up \ \} \\
 & \{f \in X \mid f.i \in K\} \in up.S \\
\equiv & \{ \ S \text{ and } T \text{ are entwined } \} \\
 & \{f \in X \mid f.i \in K\} \in T^{\#} \\
\equiv & \{ \text{ definition } T^{\#} \ \} \\
 & \forall\, B \in T : B \cap \{f \in X \mid f.i \in K\} \neq \emptyset \\
\equiv & \{ \text{ calculus } \} \\
 & \forall\, B \in T : K \cap \{g.i \mid g \in B\} \neq \emptyset \\
\equiv & \{ \text{ definition of } \# \ \} \\
 & K \in \{\{g.i \mid g \in B\} \mid B \in T\}^{\#} \ . \ \square
\end{array}
$$

Note that we do not use the Axiom of Choice in this proof.

Corollary 2. *Let X be a universally distributive ordered set. Let Z be an arbitrary set. Then the set of the functions $Z \to X$ is universally distributive.*

5 An Alternative Characterization

In this section, we prove an apparently simpler characterization of universal distributivity, which again is inspired by the Wikipedia item mentioned above.

Let a subset B of X be called *down-closed* if $x \leq y \in B$ implies $x \in B$ for all x and y. Let $D.X$ be the set of down-closed subsets of X. We claim

Theorem 5. *A complete ordered set X is universally distributive if and only if*

$$\text{(5)} \qquad \forall\, T \in \mathbb{P}.(D.X) : \ \bigwedge\{\bigvee B \mid B \in T\} = \bigvee \bigcap T \ .$$

Note that, in words, T ranges over all sets of down-closed subsets of X.

In order to prove this theorem, we need to relate arbitrary subsets of X to down-closed ones. We therefore define the down-closure function $down : \mathbb{P}.X \to D.X$ by $down.B = \{x \mid \exists\, b \in B : x \leq b\}$. We also use the associated direct image function $down^\dagger : \mathbb{P}.(\mathbb{P}.X) \to \mathbb{P}.(D.X)$. Theorem 5 is based on the following general result:

Theorem 6. *Let* $T \in \mathbb{P}.(\mathbb{P}.X)$. *Then* $\bigvee\{\bigwedge A \mid A \in T^\#\} = \bigvee \bigcap(down^\dagger.T)$.

Proof. Let us write *LHS* and *RHS* for the lefthand side and the righthand side of the equality. We prove the equality by means of two inequalities.

First, let $x \in \bigcap(down^\dagger.T)$. For any $B \in T$, we have $x \in down.B$, so there is $y \in B$ with $x \leq y$. This implies $\{y \mid x \leq y\} \in T^\#$. Since $\bigwedge\{y \mid x \leq y\} = x$, it follows that $x \leq LHS$. Since this holds for arbitrary x, this proves $RHS \leq LHS$.

Conversely, let $A \in T^\#$ be given. For any $B \in T$, there is some $x \in A \cap B$, which therefore satisfies $\bigwedge A \leq x \in B$, so that $\bigwedge A \in down.B$. This implies that $\bigwedge A \in \bigcap(down^\dagger.T)$ and hence $\bigwedge A \leq RHS$. Since this holds for arbitrary A, it follows that $LHS \leq RHS$. □

We can now prove Theorem 5. First, assume that X is universally distributive. We verify formula (5), by observing, for any $T \in \mathbb{P}.(D.X)$, that

$$\begin{array}{ll} & \bigwedge\{\bigvee B \mid B \in T\} \\ = & \{ \text{ universal distributivity } \} \\ & \bigvee\{\bigwedge A \mid A \in T^\#\} \\ = & \{ \text{ Theorem 6 and } down^\dagger.T = T \ \} \\ & \bigvee \bigcap T \ . \end{array}$$

For the converse implication, we assume that (5) holds and observe for any set $T \in \mathbb{P}.(\mathbb{P}.X)$

$$\begin{array}{ll} & \bigvee\{\bigwedge A \mid A \in T^\#\} \\ = & \{ \text{ Theorem 6 } \} \\ & \bigvee \bigcap(down^\dagger.T) \\ = & \{ \text{ formula (5) for } T := down^\dagger.T \ \} \\ & \bigwedge\{\bigvee(down.B) \mid B \in T\} \\ = & \{ \text{ relation between } down \text{ and joins } \} \\ & \bigwedge\{\bigvee B \mid B \in T\} \ , \end{array}$$

thus proving universal distributivity. This completes the proof of Theorem 5.

For completeness, we also give the dual version of Theorem 5. Let a subset A of X be called *up-closed* if $x \geq y \in A$ implies $x \in A$ for all x and y. Let $U.X$ be the set of up-closed subsets of X. Then we have

Theorem 7. *A complete ordered set* X *is universally distributive if and only if*

$$\forall\, S \in \mathbb{P}.(U.X) : \ \bigvee\{\bigwedge A \mid A \in S\} = \bigwedge \bigcap S \ .$$

6 Post-scriptum

It is well-known that the inclusion order makes the power set $\mathbb{P}.Y$ of a set Y a complete ordered set with unions for suprema and intersections for infima. Independently of the Axiom of Choice, we have:

Corollary 3. *The power set* $\mathbb{P}.Y$ *of any set* Y *is universally distributive.*

Proof. By Theorem 5, it suffices to prove, for any set T of down-closed subsets of $\mathbb{P}.Y$, that $\bigwedge\{\bigvee B \mid B \in T\} = \bigvee \bigcap T$. In $\mathbb{P}.Y$, we have unions for $\bigvee$ and intersections for $\bigwedge$. It therefore suffices to observe, for any $y \in Y$:

$$
\begin{array}{ll}
& y \in \bigwedge\{\bigvee B \mid B \in T\} \\
\equiv & \forall\, B \in T : y \in \bigvee B \\
\equiv & \forall\, B \in T : \exists\, b \in B : y \in b \\
\equiv & \{\ \text{each } B \in T \text{ is down-closed }\} \\
& \forall\, B \in T : \{y\} \in B \\
\equiv & \{\ \text{each } B \in T \text{ is down-closed }\} \\
& \exists\, a : y \in a \ \wedge\ \forall\, B \in T : a \in B \\
\equiv & \exists\, a \in \bigcap T : y \in a \\
\equiv & y \in \bigvee(\bigcap T)\ .\ \Box
\end{array}
$$

References

[BD74] R. Balbes and Ph. Dwinger. *Distributive Lattices.* University of Missouri Press, 1974.

[BW90] M. Barr and C. Wells. *Category Theory for Computing Science.* Prentice Hall International, 1990.