Dilation with diamonds

Wim H. Hesselink, whh586
Bernoulli Institute, University of Groningen

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The one-dimensional dilation of sequence s with window size w is defined as the sequence

$$D_1(w)(s) = (\lambda\, n : \max\{s(i) \mid n - w < i \leq n\}) \ .$$

Sequence s is supposed to be a finite sequence, that starts with index 0. Assume $s(n) \geq 0$ for all indices n in the domain of s. We take $s(i) = -1$ for all indices outside the domain.

The rectangular two-dimensional dilation of image m with a window of size (v, h) is defined as the image $D_2(v, h)(m)$ given by

$$D_2(v, h)(m)(y, x) = \max\{m(j, i) \mid y - v < j \leq y \ \wedge \ x - h < i \leq x\} \ .$$

Here m is regarded as a function with integer arguments, with the value -1 if one of its arguments is negative or too large.

We can regard m as a sequence of sequences ("currying") and identify $m(y, x) = m(y)(x)$. Then

$$\begin{array}{ll} & D_2(v, h)(m)(y, x) \\ = & \max\{\max\{m(j)(i) \mid x - h \leq i \leq x\} \mid y - v \leq j \leq y\} \\ = & D_1(v)(\lambda j : D_1(h)(m(j))(x))(y) \ . \end{array}$$

This reduces the computation of D_2 to computations of D_1.

Let the distance function d_1 on the plane be defined by the L^1-norm given by $||u|| = |u_1| + |u_2|$, so that $d_1(u, v) = |u_1 - v_1| + |u_2 - v_2|$. For a given grid point u and radius r, the disk $\{v \mid d_1(v, u) \leq r\}$ is a "diamond", i.e., a square with the diagonals parallel to the coordinate axes. This can be compared with the L^∞-norm given by $||u||_\infty = \max(|u_1|, |u_2|)$, with its associated distance function d_∞. Its disk $\{v \mid d_\infty(v, u) \leq r\}$ is a square with edges parallel to the coordinate axes.

We now aim at the dilation with diamonds as structural elements. This means, for given image m, radius r, and grid points u, to compute the dilation

$$(0) \qquad \mathit{dil}(r, m)(u) = \max\{m(v) \mid v : d_1(u, v) \leq r\}.$$

To visualize the geometry, the coordinate u_1 or y is considered vertical and downward, and u_2 or x is considered horizonal and growing to the right. The positive quadrant therefore lies to the south-east of the origin.

The linear transformation A of the plane given by $A(y, x) = (y + x, -y + x)$ transforms the L^1 into the L^∞ norm because it satisfies

$$(1) \qquad ||A(u)||_\infty = ||u||_1$$

because of

$$\begin{array}{l} ||A(y, x)||_\infty = \max(|y + x|, |-y + x|) \\ = \max\{y + x, -y - x, -y + x, y - x\} \\ = \max(y, -y) + \max(x, -x) = |y| + |x| = ||(y, x)||_1 \ . \end{array}$$

In fact, A is a clockwise rotation over an angle of 45%, followed by a multiplication with a factor of $\sqrt{2}$, to keep the grid invariant. Next, a translation is introduced to keep the sequences within the positive quadrant.

For the latter purpose, assume that image m is contained in a rectangle R of size $N \times M$, i.e., satisfies

$$m(y,x) \neq -1 \;\Rightarrow\; 0 \leq y \leq N \;\wedge\; 0 \leq x \leq M \; .$$

Here inequalities $\leq$ are used to have the vertices in the image. Indeed, the vertices of rectangle R are, clockwise, $(0,0)$, $(0,M)$, (N,M), $(N,0)$.

Now use the transformation F that consists of application of A followed by a translation over $(0,N)$. Then F is given by

$$F(y,x) = (x+y, x-y+N) \; .$$

The transformed rectangle $F(R)$ has the vertices $(0,N)$, $(M,M+N)$, $(M+N,M)$, $(N,0)$. A straightforward calculation shows that

$$(q,p) \in F(R) \;\Rightarrow\; -N \leq q-p \leq N \;\wedge\; N \leq q+p \leq 2\cdot M + N \; . \tag{2}$$

It follows that $F(R) \subseteq R'$ where R' is the square that consists of the grid points (q,p) with $0 \leq q \leq M+N \;\wedge\; 0 \leq p \leq M+N$.

Function F satisfies $d_1(u,v) = d_\infty(F(u),F(v))$ because of formula (1). This implies that the diamond around (y,x) with radius r is transformed into the square around $F(y,x)$ with L^∞-radius r. We can therefore apply the rectangular two-dimensional dilation to the transformed image, followed by a transformation backward.

The only complication is that, roughly speaking, half of the grid points of the transformed diamond are not images of grid points. Indeed, let a grid point (y,x) be called *even*, or *odd*, iff $y+x$ is even, or odd, respectively. If N is even (or odd), function F transforms all grid points into even (odd) ones.

The program therefore first ensures that all grid points of square R' that are not transformed grid points of rectangle R hold the lowest possible value -1. It then transforms the image m into m', applies the rectangular dilation D_2 of m' with squares of size $2\cdot r+1$ as structural elements, and finally transforms backward, with a translation along (r,r) to get the dilation value d in the centers of the diamonds.

(3) **for each** $u \in R'$ **do** $m'(u) := -1$ **endwhile** ;
for each $v \in R$ **do** $m'(F(v)) := m(v)$ **endwhile** ;
$d' = D_2(2\cdot r+1, 2\cdot r+1)(m')$;
for each $v \in R$ **do** $d(v) := d'(F(v)+(r,r))$ **endwhile** .

It remains to remove as much redundant computation as possible. Firstly, the computation of D_2 can be restricted to the transformed rectangle $F(R)$ with its bounds given by formula (2). This is a reduction of $(N+M)^2$ grid points to $2\cdot N\cdot M$ grid points. When this has been done the same reduction can be applied to the first line of (3).